Strong convergence and asymptotic exponential stability of modified truncated EM method for neutral stochastic differential equations with time-dependent delay

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0. Outline:

1. Backgrounds

Neutral differential delay equation was firstly introduced by Brayton [Brayton, R., Quat. Appl. Math. 1976, 24, 289-301.]. He used a partial differential equation (PDE) to describe the problem of loseless transmission and then transferred the PDE into the following NDDE

$$\frac{d}{dt}[x(t) - Kx(t-\tau)] = f(x(t), x(t-\tau))$$

Rubanik [Rubanik, V.P. Oscillations of Quasilinear Systems with Retardation; Nauka: Moscow, 1969.] considered this type equation in his study of vibrating masses attached to an elastic bar.
In general, an NDDE has the form

$$\frac{d}{dt}[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), t)$$

Taking the environmental disturbances into account, Kolmanovskii and Nosov [Kolmanovskii, V.B.; Nosov, V.R. Nauka: Moscow, 1981.] and Mao [Mao X. Stochastic Differential Equations and Their Applications (2nd edition); Horwood Pub.: Chichester, 2007.] discussed the neutral stochastic differential delay equations (NSDDEs)

$$\begin{aligned} d(x(t) - D(x(t-\tau), t)) = & f(x(t), x(t-\tau), t)dt \\ & + g(x(t), x(t-\tau), t)dB(t) \end{aligned} \tag{1}$$

Sometimes the delay might be time-dependent:

$$\begin{aligned} d(x(t) - D(x(t-\delta(t)), t)) = & f(x(t), x(t-\delta(t)), t)dt \\ & + g(x(t), x(t-\delta(t)), t)dB(t) \end{aligned} \tag{2}$$

Considering time-dependent delay has practical meaning because many systems depend not only on the past states of their own, the delay might not be constant. For example, in the pandemic of COVID-19, when they are infected, many people will have the symptoms after different period of time, and as time goes by, the latency of the virus becomes longer and longer, then the number of people with symptoms can only be described by a system with time-dependent delay, so the systems with time-dependent delays become more and more important.

However, the exact solution to the equation is usually difficult to obtain even the existence and uniqueness holds. So numerical simulations become more and more important to investigate the properties of exact solutions.

Strong convergence is a major topic for the given numerical simulation. Indeed, if it does not converge to the exaction, it will be useless even it have some good properties.

Asymptotic stability is also an important topic. Roughly speaking, the stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. So it is interesting to investigate the sufficient conditions under which the numerical simulation replicates the asymptotic stability of the exact solution.

In recent years, convergence and asymptotic stability of the numerical methods for stochastic differential delay equations (SDDEs) have been discussed intensively by many researchers. For example,

1.Gan et. al [S. Gan, H. Schurz, H. Zhang, Int. J. Numer. Anal. Model. Ser. B, 8(2011) 201-213.] investigated mean-square convergence of stochastic θ method under global Lipschitz condition,

2.[H. Zhang, S. Gan, Appl. Math. Comput., 204 (2008) 884-890.] studied L^2 convergence of one-step methods under the same assumptions,

3. [L. Tan, C. Yuan, Bull. Math. Sci., 9(2019)1950006.] considered convergence of theta-method for a class of SDDEs under local Lipschitz condition,

4.Zhang et.al [W. Zhang, M. Song, M. Liu, J. Comput. Appl. Math., 335 (2018) 114-128.] considered strong convergence of the partially truncated Euler-Maruyama method for a class of stochastic differential delay equations,

5. Lan and Wang [G. Lan, Q. Wang, J. Comput. Appl. Math., 362(2019)83-98.] obtain the sufficient conditions of strong convergence of (MTEM) method for NSDDEs.
6. [L. Liu,Q. Zhu, J. Comput. Appl. Math. 305 (2016) 55-67.] consider the mean-square stability of theta method,
7.[H. Mo, X. Zhao, F. Deng, Math. Methods Appl. Sci. 40 (2017) 1794-1803.] consider mean-square stability of the backward Euler-Maruyama method,
8.[W. Wang, Y. Chen, Appl. Numer. Math. 61 (2011) 696-701.] investigated mean-square stability of semi-implicit Euler method for nonlinear NSDDEs.

We point out that all the above mentioned papers only consider constant delay case.

For the time-dependent delay case, see e.g.
1.[C. Yue, L. Zhao, J. Comput. Appl. Math.,382(2021)113087.] considers strong convergence of the split-step backward Euler method for SDDE with nonlinear diffusion coefficient,
2.[Q. Zhu, Systems Control Lett., 118(2018)62-68.] considers the stability of stochastic delay differential equations with Lévy noise when the delay is bounded,
3. [M. Milošević, Math. Comput. Modelling 57(2013)] and [M. Obradović, M. Milošević, J. Comput. Appl. Math., 309 (2017) 244-266.] investigate stability of exact solution and the corresponding Euler-Maruyama method for NSDEs with bounded delay, moreover, the drift term f must satisfy linear growth condition.

However, to the best of our knowledge, few results consider the exponential stability of the numerical methods for NSDEs with unbounded delay.

In this talk, we will first define MTEM method for NSDEs with time-dependent delay, and then we will investigate the strong convergence of MTEM method and asymptotic exponential stability of the exact solution and the corresponding MTEM method.

2. Settings and truncated Euler-Maruyama method

Let $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0}, P)$ be a complete filtered probability space. Let $\tau \geq 0$ be a constant and denote by $C_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family $\mathscr{F}_0$ measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables. We consider equation (2) with the initial value

$$x_0 = \xi = \{\xi(\theta), \theta \in [-\tau, 0]\} \in C_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n),$$

where $B(t)$ is a d-dimensional standard Brownian motion,
$\delta(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $\delta(0) = \tau$,
$||\xi||_p := (\mathbb{E} \sup_{\theta\in[-\tau,0]} |\xi(\theta)|^p)^{\frac{1}{p}} < \infty$, moreover,
$f : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n \otimes \mathbb{R}^d$ are Borel measurable functions, respectively.

Throughout this talk, let

$$|\delta'(t)| \leq \eta < 1. \tag{3}$$

Local Lipschitz condition on f, g:
For each R, there is $L_R > 0$

$$\begin{aligned} &|f(x,y,t) - f(\bar{x},\bar{y},t)| \vee |g(x,y,t) - g(\bar{x},\bar{y},t)| \\ &\leq L_R\left(|x-\bar{x}| + |y-\bar{y}|\right) \end{aligned} \tag{4}$$

for all $t \geq 0, |x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq R$.

Global Lipschitz condition on D:
There exist $0 < u < 1$ and U such that

$$|D(x,t) - D(y,s)| \leq u|x-y| + U|t-s|. \tag{5}$$

Let $\Delta \in (0,1)$. And for the given $\tau > 0$, there exists a positive integer m such that $\tau = m\Delta$ (if $\tau = 0$, we can choose any $\Delta > 0$ small enough).

Choosing $\Delta^* > 0$ which is small enough, let $h(\Delta)$ be a strictly positive decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that for some sufficiently small $0 < \varepsilon < 1$,

$$\lim_{\Delta \to 0} h(\Delta) = \infty. \lim_{\Delta \to 0} L^4_{h(\Delta)} \Delta^{1-\varepsilon} = 0. \tag{6}$$

Such h always exists for given L_R.

Then we will define modified truncated function f_Δ for any $\Delta > 0$

$$f_\Delta(x,y,t) = \begin{cases} f(x,y,t), |x| \vee |y| \le h(\Delta), \\ \frac{|x|\vee|y|}{h(\Delta)} f(\frac{h(\Delta)}{|x|\vee|y|}x, \frac{h(\Delta)}{|x|\vee|y|}y, t), |x| \vee |y| > h(\Delta). \end{cases} \quad (7)$$

g_Δ is defined in the same way as f_Δ.

Mao [X. Mao, J. Comput. Appl. Math. 290 (2015) 370-384] first introduced truncated function $f_\Delta(x) = f((\frac{h(\Delta)}{|x|} \wedge 1)x)$ of a given function and the truncated Euler-Maruyama method, and then generalized it to the SDDEs case.

Modified truncated Euler-Maruyama method

Now, the modified truncated EM (MTEM) method $X_k^\Delta \approx x(k\Delta)$ can be defined by setting $X_k^\Delta = \xi(k\Delta)$ for every integer $k = -m, \cdots, 0$ and for $k = 1, 2, \cdots$

$$\begin{aligned} X_{k+1} = D(X_{k+1-I_{k+1}}, (k+1)\Delta) + X_k - D(X_{k-I_k}, k\Delta) \\ + f_\Delta\left(X_k, X_{k-I_k}, k\Delta\right)\Delta + g_\Delta\left(X_k, X_{k-I_k}, k\Delta\right)\Delta B_k \end{aligned} \tag{8}$$

where $I_k = [\frac{\delta(k\Delta)}{\Delta}]$ and $[x]$ is the integer part of x and $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$.

The continuous-time MTEM solutions are as follows:

$$\bar{x}_\Delta(t) = \sum_{k=-m}^{\infty} X_k 1_{[k\Delta,(k+1)\Delta)}(t), \tag{9}$$

$$\tilde{x}_\Delta(t) = \sum_{k=-m}^{\infty} X_{k-[\frac{\delta(k\Delta)}{\Delta}]} 1_{[k\Delta,(k+1)\Delta)}(t) \tag{10}$$

and $x_\Delta(t) = \xi(t), t \in [-\tau, 0]$, for $t \geq 0$,

$$\begin{aligned} x_\Delta(t) = {} & D(\tilde{x}_\Delta(t), t) + \xi(0) - D(\xi(-\tau), 0) \\ & + \int_0^t f_\Delta(\bar{x}_\Delta(s), \tilde{x}_\Delta(s), s)ds + \int_0^t g_\Delta(\bar{x}_\Delta(s), \tilde{x}_\Delta(s), s)dB(s). \end{aligned} \tag{11}$$

It is obvious that $x_\Delta(k\Delta) = \bar{x}_\Delta(k\Delta) = X_k$, for all $k \geq 0$.

Strong convergence of MTEM for (2)

3.1 Existence and uniqueness of the exact solution:
There exist positive contants K, l and $p \geq 2$ such that

$$2\langle x - D(y,t), f(x,y,t)\rangle + (p-1)|g(x,y,t)|^2 \leq K(1+|x|^2+|y|^2). \tag{12}$$

Then we have
Theorem 1: Suppose (4), (5) and (12) hold. Then for any fixed $T > 0, p \geq 2$ and initial value ξ satisfying $||\xi||_p < \infty$, (1) has a unique global solution $x(t)$ on $[0, T]$ satisfying $\sup_{0\leq t\leq T} \mathbb{E}|x(t)|^p \leq M$. Denote $\tau_R = \inf\{t \geq 0, |x(t)| \geq R\}$. Then $P(\tau_R \leq T) \leq \frac{C}{R^p}$.
Corollary 1: Let (4), (5), (12) and $|g(x,y,t)|^2 \leq \bar{K}(1+|x|^r+|y|^r)$ hold for $p > 2$ and $2 < r < p$. Set $\bar{p} = p - r + 2$. If $||\xi||_p < \infty$, then $\mathbb{E}(\sup_{0\leq t\leq T} |x(t)|^{\bar{p}}) \leq C$.

3.2 Some useful lemmas

Now let us consider the moment boundedness of $\sup_{0\le t\le T} x_\Delta(t)$, we need a stronger condition:

There exists positive constant K such that $\forall x, y \in \mathbb{R}^d, a \in (0,1]$,

$$2\left\langle x - aD(\frac{1}{a}y, t), f(x,y,t)\right\rangle + (p-1)|g(x,y,t)|^2 \le K(1+|x|^2+|y|^2). \tag{13}$$

Lemma 1 Suppose that (4), (5), (13) and (6) hold. If

$$|g(x,y,t)|^2 \le \bar{K}(1+|x|^r+|y|^r)$$

holds for $p > 2$ and $2 < r < p$ and the initial value satisfies $||\xi||_p < \infty$, then there exist sufficiently small $\Delta^* > 0$ and $C > 0$ such that

$$\sup_{0<\Delta\le\Delta^*} \mathbb{E}(\sup_{0\le t\le T} |x_\Delta(t)|^{\bar{p}}) \le C, \forall T > 0.$$

Define $\rho_{\Delta,R} = \inf\{t \ge 0, |x_\Delta(t)| \ge R\}$. Then $P(\rho_{\Delta,R} \le T) \le \frac{C}{R^p}$.

Lemma 2 Suppose that all assumptions of Lemma 1 hold. Then for any $\Delta \in (0, \Delta^*)$ and $0 < \varepsilon < 1$, there exists $C > 0$ which is independent of Δ (but dependent on n) such that

$$\mathbb{E} \sup_{0 \le t \le T} |x_\Delta(t) - \bar{x}_\Delta(t)|^q \le C L_{h(\Delta)}^q \Delta^{\frac{q(1-\varepsilon)}{2}}. \tag{14}$$

The proof of the above two lemmas is similar to [L., Wang(2019)].

3.3 Convergence in the ball.

Now suppose that there exists $\hat{K} > 0$ such that for any fixed Δ the initial value satisfies

$$\mathbb{E} \sup_{-m \le k \le -1} \sup_{k\Delta \le s \le (k+1)\Delta} |\xi(s) - \xi(k\Delta)|^q \le \hat{K}\Delta^{\frac{q}{2}}. \qquad (15)$$

Lemma 3 Suppose that all assumptions of Lemma 2 and (15) hold. Set

$$\theta_{\Delta,R} = \tau_R \wedge \rho_{\Delta,R} \quad \textit{and} \quad e_\Delta(t) = x(t) - x_\Delta(t) \text{ for } t \ge 0.$$

Then for any $\Delta \in (0, \Delta^*)$, $0 < \varepsilon < 1$ and $R \le h(\Delta^*)$, there exists $C(q, T) > 0$ (independent of Δ) such that

$$\mathbb{E}(\sup_{0 \le t \le T} |e_\Delta(t \wedge \theta_{\Delta,R})|^q) \le C(q, T) L_{h(\Delta)}^{2q} \Delta^{\frac{q(1-\varepsilon)}{2}}.$$

Critical point:

Notice that since $\delta' \le \eta < 1$, then for $s \in [k\Delta, (k+1)\Delta)$, we have

$$s - \delta(s) \in [k\Delta - \delta(k\Delta), (k+1)\Delta - \delta((k+1)\Delta))\,.$$

Moreover,

$$s - \delta(s) \in \left[(k-1)\Delta - [\frac{\delta(k\Delta)}{\Delta}]\Delta, (k+1)\Delta - [\frac{\delta((k+1)\Delta)}{\Delta}]\Delta\right)$$

and

$$\begin{aligned}
&(k+1)\Delta - [\frac{\delta((k+1)\Delta)}{\Delta}]\Delta - ((k-1)\Delta - [\frac{\delta(k\Delta)}{\Delta}]\Delta) \\
&= 2\Delta + \left([\frac{\delta(k\Delta)}{\Delta}] - [\frac{\delta((k+1)\Delta)}{\Delta}]\right)\Delta \\
&\le 2\Delta + \left(\left[\frac{\delta(k\Delta) - \delta((k+1)\Delta)}{\Delta}\right] + 1\right)\Delta \\
&\le 2\Delta + \left(\left[\frac{|\delta'(\theta)|\Delta}{\Delta}\right] + 1\right)\Delta \\
&\le 3\Delta
\end{aligned}$$

Thus $\bar{z}_\Delta(s-\delta(s)) = Z_{k-1-[\frac{\delta(k\Delta)}{\Delta}]}, Z_{k-[\frac{\delta(k\Delta)}{\Delta}]}$ or $Z_{k+1-[\frac{\delta(k\Delta)}{\Delta}]}$. It follows that

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}|\bar{z}_\Delta(s-\delta(s)) - \tilde{z}_\Delta(s)|^q \le{}& \mathbb{E}\sup_{0\le k\le[\frac{t}{\Delta}]}|Z_{k+1} - Z_k|^q \\
&+ \mathbb{E}\sup_{-m\le k\le -1}\ \sup_{k\Delta\le s\le(k+1)\Delta}|\xi(s) - \xi(k\Delta)|^q.
\end{aligned}$$

3.3 Strong convergence of MTEM method.

By a standard procedure of truncation, we have

Theorem 2 Suppose all assumptions in Lemma 3 hold. If $|g(x,y)|^2 \leq \bar{K}(1+|x|^r+|y|^r), \forall x,y \in \mathbb{R}^d$ holds for some r $(2 \leq r \leq (p-2), 2 \leq q \leq (p-r))$, then for any $0 < \varepsilon < 1$, there exists $C(q,T)$ (independent of Δ) such that

$$\mathbb{E} \sup_{0 \leq t \leq T} |x(t) - x_\Delta(t)|^q \leq C(q,T) L_{h(\Delta)}^{2q} \Delta^{\frac{q(1-\varepsilon)}{2}}. \qquad (16)$$

Asymptotic exponential stability of solution of (2)

We always assume that

$$f(0,0,t) \equiv 0, \quad g(0,0,t) \equiv 0, \quad D(0,t) \equiv 0$$

which implies that $x \equiv 0$ is the trivial solution of equation (1).
We need a stronger Lipschitz condition on D.
There exists constants $l > 0$ and $0 < u < 1$ such that

$$|D(x,t) - D(y,t)| \leq u(1+l)^{-\frac{\delta(t)}{2}} |x - y| \tag{17}$$

and suppose there exist positive constants K, l such that

$$2\langle x - D(y,t), f(x,y,t)\rangle + |g(x,y,t)|^2 \leq -\lambda_1 |x|^2 + \lambda_2 (1+l)^{-\delta(t)} |y|^2. \tag{18}$$

Then we have

Theorem 3 Assume that (4), (17) and (18) hold with $\lambda_1 > \frac{\lambda_2}{1-\eta}$. Then for any initial condition $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$, the exact solution of equation of (2) is mean-square exponential stable. That is, there exists $\lambda > 0$ such that $\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t)|^2}{t} \leq -\lambda$.

Asymptotic exponential stability of the MTEM method

We need the following lemma (see [M. Milošević, Math. Comput. Modelling 57(2013)])
Lemma 4 Suppose (3) holds, for any fixed $i \in \{1, 2, ...\}$, denote $i - [\frac{\delta(i\Delta)}{\Delta}] = n \in \{-m, -m+1, ..., 0, 1, ..., i\}$, then

$$\#\{j \in 0, 1, 2, ... : j - [\frac{\delta(j\Delta)}{\Delta}] = n\} \leq [(1-\eta)^{-1}] + 1. \qquad (19)$$

Here $\#A$ is the number of elements of the set A.

Exponential stability of X_k

Now we need a Khasminskii type condition:
There exist positive constants K, l such that

$$2\left\langle x - aD(\frac{y}{a}, t), f(x, y, t)\right\rangle + |g(x, y, t)|^2 \le -\lambda_1 |x|^2 + \lambda_2 (1+l)^{-\delta(t)} |y|^2 \tag{20}$$

for any $x, y \in \mathbb{R}^d$, $a \in (0, 1]$ and a stronger local Lipschitz condition on f:
There exists $l > 0$ such that for each R, there is $L_R > 0$

$$|f(x, y, t) - f(\bar{x}, \bar{y}, t)| \le L_R \left(|x - \bar{x}| + (1 + l)^{-\frac{\delta(t)}{2}} |y - \bar{y}| \right) \tag{21}$$

for all $t \ge 0$, $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \le R$.
Now we have
Theorem 4 Suppose (17), (20) and (21) hold with $\lambda_1 > \lambda_2([(1-\eta)^{-1}] + 1)$. Then for $h(\Delta)$ satisfying $h(\Delta) \to \infty$ and $L^2_{h(\Delta)}\Delta \to 0$ as $\Delta \to 0$, X_k is mean-square exponentially stable.

Theorem 4 contains both bounded and unbounded delay cases. In the constant delay case, we have shown in [Lan, J. Comput. Appl. Math., 340(2018) 334-341] that the MTEM method is exponentially stable under the following condition:
There exist positive constants $\lambda_1 > \lambda_2$ such that

$$2\left\langle x - aD(\frac{1}{a}y, t), f(x, y, t)\right\rangle + |g(t, x, y)|^2 \le -\lambda_1|x|^2 + \lambda_2|y|^2 \quad (22)$$

for $\forall x, y \in \mathbb{R}^d, a \in (0, 1]$.
Notice that we can choose sufficiently small $\varepsilon > 0$ and $l > 0$ such that $\lambda_1 > \lambda_2 + \varepsilon$

$$\lambda_2 \le (\lambda_2 + \varepsilon)(1 + l)^{-\tau} \le (\lambda_2 + \varepsilon)(1 + l)^{-\delta(t)}. \quad (23)$$

So (20) holds, and then Theorem 4 covers the stability result of the MTEM method for constant delay case.

Sketch of proof:

$$\mathbb{E}|Y_{k+1}|^2 \le \mathbb{E}|Y_k|^2 - \Delta(\lambda_1 - \varepsilon')\mathbb{E}|X_k|^2 + \Delta(\lambda_2 + \varepsilon')(1+l)^{-\delta(k\Delta)}\mathbb{E}|X_{k-[\frac{\delta(k\Delta)}{\Delta}]}|^2,$$

where $Y_k = X_k - D(X_{k-[\frac{\delta(k\Delta)}{\Delta}]}, k\Delta)$.

$$\begin{aligned}
&C^{(k+1)\Delta}\mathbb{E}|Y_{k+1}|^2 - C^{k\Delta}\mathbb{E}|Y_k|^2 \\
&\le -\Delta(\lambda_1 - \varepsilon')C^{(k+1)\Delta}\mathbb{E}|X_k|^2 + \Delta(\lambda_2 + \varepsilon')C^{(k+1)\Delta}(1+l)^{-\delta(k\Delta)}\mathbb{E}|X_{k-[\frac{\delta(k\Delta)}{\Delta}]}|^2 \\
&\quad + (C^{(k+1)\Delta} - C^{k\Delta})\mathbb{E}|Y_k|^2 \\
&\le -\Delta(\lambda_1 - \varepsilon')C^{(k+1)\Delta}\mathbb{E}|X_k|^2 + \Delta(\lambda_2 + \varepsilon')C^{(k-[\frac{\delta(k\Delta)}{\Delta}]+1)\Delta}\mathbb{E}|X_{k-[\frac{\delta(k\Delta)}{\Delta}]}|^2 \\
&\quad + (C^{(k+1)\Delta} - C^{k\Delta})\mathbb{E}|Y_k|^2.
\end{aligned}$$

$$\begin{aligned}
&C^{k\Delta}\mathbb{E}|X_k|^2 \\
&\le (1+c_2)C^{k\Delta}\mathbb{E}|Y_k|^2 + C^{k\Delta}(1+c_2^{-1})u^2(1+l)^{-\delta(k\Delta)}\mathbb{E}|X_{k-[\frac{\delta(k\Delta)}{\Delta}]}|^2 \\
&\le (1+c_2)D + u^2(1+c_2^{-1})\left(\frac{C}{1+l}\right)^{\delta(k\Delta)} C^{(k-[(\frac{\delta(k\Delta)}{\Delta}])\Delta}\mathbb{E}|X_{k-[\frac{\delta(k\Delta)}{\Delta}]}|^2.
\end{aligned}$$

Numerical example

Example 1 For any fixed $\varepsilon \in (0,1)$, consider the scalar NSDDE

$$d\left(x(t) - \frac{1}{2}\sin(t)\cdot \sin x\left(\frac{t}{2}-1\right)\right) = \left(x(t) - x(t)|x|^{2\varepsilon}(t) + \sin x\left(\frac{t}{2}-1\right)\right)dt + \sin(t)\frac{|x|^{1+\varepsilon}(t) + x\left(\frac{t}{2}-1\right)}{4}dB_t \tag{24}$$

with initial value $\xi(\theta) \equiv 1, \theta \in [-1,0]$. Here $f(x,y,t) = x - x|x|^{2\varepsilon} + \sin y$, $g(x,y,t) = \sin(t)\frac{|x|^{1+\varepsilon}+y}{4}$ and $D(y,t) = \frac{1}{2}\sin(t)\sin y$ and $\delta(t) = 1 + \frac{t}{2}$.

Obviously, f and g satisfy local Lipschitz condition (4) with $L_R = (1+2\varepsilon)(1+R^{2\varepsilon})$, (5) holds with $u = U = \frac{1}{2}$ and (13) holds with $p = 6$ and $K = \frac{7}{2}$. Moreover, $|g(x,y)|^2 \leq \frac{1}{8}(1+|x|^{2+2\varepsilon}+|y|^{2+2\varepsilon})$, i.e. $r = 2(1+\varepsilon)$. For $\varepsilon' > 0$ small enough, define

$$h(\Delta) = \left(\frac{\Delta^{-\frac{\varepsilon'}{4}}}{1+2\varepsilon} - 1\right)^{\frac{1}{2\varepsilon}}, \quad \Delta < (1+2\varepsilon)^{-\frac{4}{\varepsilon'}}.$$

Then $h(\Delta) \to \infty$ as $\Delta \to 0$ and for if $1-\varepsilon-\varepsilon' > 0$, we have $L^4_{h(\Delta)}\Delta^{1-\varepsilon} = \Delta^{1-\varepsilon-\varepsilon'} \to 0$ as $\Delta \to 0$. That is, (6) holds for this h.

Choose $q = 3$ and $\frac{\varepsilon}{\frac{1}{4}+\varepsilon} < \varepsilon' < 1 - \varepsilon$. Then for sufficiently small Δ, $h(\Delta) \sim \Delta^{-\frac{\varepsilon'}{8\varepsilon}} \geq \Delta^{\frac{\varepsilon'-1}{2}}$. Therefore, we can choose $C' > 0$ such that

$$\begin{aligned} C'(L_{h(\Delta)}^{2q}\Delta^{q(1-\varepsilon)})^{-\frac{1}{p-q}} &= C'(L_{h(\Delta)}^{4}\Delta^{1-\varepsilon})^{-\frac{1}{2}} \\ &= C'\Delta^{\frac{\varepsilon'+\varepsilon-1}{2}} \leq C'\Delta^{\frac{\varepsilon'-1}{2}} \leq h(\Delta), \end{aligned}$$

Then by Theorem 2, we have

$$\mathbb{E}\sup_{0\leq t\leq T}|x(t) - x_\Delta(t)|^3 \vee \mathbb{E}\sup_{0\leq t\leq T}|x(t) - x_\Delta(t)|^3 \leq C\Delta^{\frac{3(1-\varepsilon-\varepsilon')}{2}}.$$

Example 2 Let $d = 1$. Consider the following scalar NSDDE:

$$d\left(x(t) - \frac{1}{6}e^{-\frac{t}{2}}\sin x\left(\frac{t}{2}\right)\right) = \left(-6x(t) - x^3(t) + \frac{1}{2}e^{-\frac{t}{2}}\sin x\left(\frac{t}{2}\right)\right)dt + \left(\frac{x^2(t)}{2} + \frac{1}{2}e^{-\frac{t}{4}}x\left(\frac{t}{2}\right)\right)dB_t. \tag{25}$$

In this case, $f(x, y, t) = -6x - x^3 + \frac{1}{2}e^{-\frac{t}{2}}\sin y$, $g(x, y, t) = \frac{x^2 + ye^{-\frac{t}{4}}}{2}$, $D(y, t) = \frac{1}{6}e^{-\frac{t}{2}}\sin y$ and the delay term $\delta(t) = \frac{t}{2}$.

We can derive that

$$|D(x,t) - D(y,t)| \leq \frac{1}{6}e^{-\frac{t}{2}}|\sin x - \sin y| \leq \frac{1}{6}e^{-\frac{t}{2}}|x-y|$$

and for any $a \in (0,1]$

$$\begin{aligned}
&2\langle x - aD(\frac{1}{a}y,t), f(x,y,t)\rangle + |g(x,y,t)|^2 \\
&= -12x^2 - 2x^4 + x\sin y e^{-\frac{t}{2}} + 2ax\sin\frac{y}{a}e^{-\frac{t}{2}} + \frac{x^3 a}{3}\sin\frac{y}{a}e^{-\frac{t}{2}} \\
&\quad - \frac{a}{6}\sin y \sin\frac{y}{a}e^{-t} + (\frac{x^2 + ye^{-\frac{t}{4}}}{2})^2 \\
&\leq -12x^2 - 2x^4 + |xy|e^{-\frac{t}{2}} + |2ax\cdot\frac{y}{a}|e^{-\frac{t}{2}} \\
&\quad + \frac{1}{4}x^4 + \frac{a^4}{12}(\sin^4\frac{y}{a})e^{-t} + \frac{a}{6}|y\cdot\frac{y}{a}|e^{-\frac{t}{2}} + x^4 + e^{-\frac{t}{2}}y^2 \\
&\leq -12x^2 + 3\frac{x^2 + y^2e^{-t}}{2} + (\frac{1}{12} + \frac{1}{6} + 1)y^2e^{-\frac{t}{2}} \\
&\leq -\frac{21}{2}x^2 + 3y^2e^{-\frac{t}{2}}.
\end{aligned}$$

Moreover, mean value theorem implies that

$$
\begin{aligned}
&|f(x,y,t) - f(\bar{x},\bar{y},t)| \vee |g(x,y,t) - g(\bar{x},\bar{y},t)| \\
&\le 6(1+R^2)(|x-\bar{x}| + e^{-\frac{t}{2}}|y-\bar{y}|)
\end{aligned}
$$

for $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \le R$, $R > 0$ and $t \ge 0$.
Thus conditions in Theorem 5 hold for
$u = \frac{1}{6}, l = e-1, \eta = \frac{1}{2}, \lambda_1 = \frac{21}{2} > 9 = 3 \times 3 = \lambda_2 \times ([(1-\eta)^{-1}]+1)$ and $L_R = 6(1+R^2)$. So the MTEM method X_k is mean-square exponentially stable (therefore almost sure exponentially stable). Moreover, since (17), (20) and (21) imply (4), (5) and (18), by Theorem 3, the exact solution $x(t)$ of equation (1) is exponentially stable. Now let $h(\Delta) = \Delta^{-1/8}, \Delta < 1$. Then $1 < h(\Delta) \to \infty$ as $\Delta \to 0$, moreover, we have

$$
L^2_{h(\Delta)}\Delta = 36(1+h^2(\Delta))^2\Delta = 36(1+\Delta^{-1/4})^2\Delta \to 0 \text{ as } \Delta \to 0.
$$

Therefore, the corresponding MTEM method X_k replicates the mean-square exponential stability of given NSDDE.

Let $x(0) \equiv 2$, $\Delta = 0.1$, then computer simulation (Matlab) for the first 500 steps of X_k indicates the almost surely exponential stability, the following Figure illustrates that the numerical approximation X_k is stable and $\frac{\log |X_k|}{k\Delta}$ is less than -1 for k large enough and therefore X_k is exponentially stable.

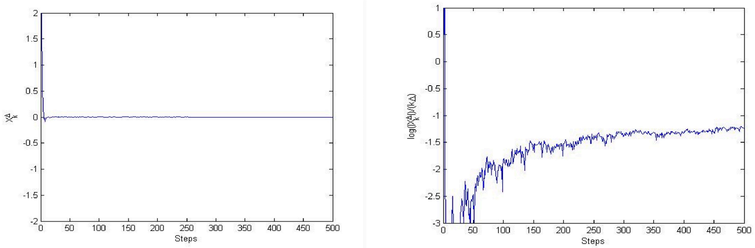

Figure: trajectory of X_k and $\frac{\log |X_k|}{k\Delta}$

Thanks!